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On the degree of commutativity of p-groups of maximal class ☆

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A B S T R A C T

The first major study of p-groups of maximal class was made by Blackburn in 1958. He showed that an important invariant parameter of these groups is its 'degree of commutativity', which is a measure of the commutativity among the members of the lower central series of G.

In this paper, we find a lower bound for the degree of commutativity of some p-groups of maximal class of order p^m, by using m and another two known invariant parameters of a p-group of maximal class. This bound improves the best one given in terms of the order of G.

1. Introduction

A group G of order p^m, with $m \geq 4$, is said to be a p-group of maximal class if $Y_{m-1} \neq 1$, where $Y_0 = G$, $Y_i = [\overbrace{G, \ldots, G}^{i}]$ for every $i \geq 2$ and Y_1 is such that $Y_1/Y_4 = C_{G/Y_4}(Y_2/Y_4)$.

The most important invariant parameter of a p-group of maximal class G is its degree of commutativity. It was introduced by Blackburn (cf. [1]) and it is defined by

$$c(G) = c = \max\{k \leq m-2 \mid [Y_i, Y_j] \leq Y_{i+j+k}, \forall i, j \geq 1\}.$$

We denote the residual class of $c(G) = c$ modulo $p-1$ by $c_0(G) = c_0$.

Following Blackburn's ideas (cf. [1]), we choose a pair of elements $s \in G \setminus \big(Y_1 \cup C_G(Y_{m-2})\big)$ and $s_1 \in Y_1 \setminus Y_2$, and we define recursively $s_i = [s_{i-1}, s] \in Y_i \setminus Y_{i+1}$ for $i = 2, \ldots, m-1$. For $i + j \leq m - c - 1$, let $\alpha_{i,j}(G) = \alpha_{i,j} \in \mathbb{F}_p$ be determined by the congruence

$$[s_i, s_j] \equiv s_{i+j+c}^{\alpha_{i,j}} \pmod{Y_{i+j+c+1}}.$$

It is known that $\alpha_{i,j}$ satisfies the following properties:

(P1) $\alpha_{i,j} = -\alpha_{j,i}$.
(P2) $\alpha_{i,i} = 0$ if $2i \leq m - c - 1$.
(P3) $\alpha_{i,j} = \alpha_{i+1,j} + \alpha_{i,j+1}$ if $i + j + 1 \leq m - c - 1$ (Bernoulli's property).
(P4) $\alpha_{i,j} = \alpha_{i+p-1,j} = \alpha_{i,j+p-1}$ if $i + j + p - 1 \leq m - c - 1$ (periodicity modulo $p - 1$).
(P5) $f(i,j,k) = \alpha_{i,j}\alpha_{i+j+c,k} + \alpha_{j,k}\alpha_{j+k+c,i} + \alpha_{k,i}\alpha_{k+i+c,j} = 0$ for any positive integers i, j, k satisfying $i + j + k \leq m - 2c - 1$ (Jacobi's identity).

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From (P2) and (P3), it is easy to check that $\alpha_{i,i+1} = \alpha_{i,i+2}$ if $2i + 2 \leq m - c - 1$. We denote $\alpha_{i,i+1}$ by x'_i. We can write $\alpha_{i,j}$ in terms of the x'_k (cf. [7]):

$$\alpha_{i,j} = \sum_{k=i}^{\left[\frac{i+j-1}{2}\right]} (-1)^{k-i} \binom{j-k-1}{k-i} x'_k. \tag{1}$$

We put the α's in a table τ_G like this one:

$$\tau_G = \begin{cases} \alpha_{1,m-c-2} & \alpha_{2,m-c-3} & \cdots & \alpha_{\left[\frac{m-c-3}{2}\right],m-c-1-\left[\frac{m-c-3}{2}\right]} \\ \vdots & \vdots & \iddots & \\ \alpha_{1,6} & \alpha_{2,5} & \alpha_{3,4} & \\ \alpha_{1,5} & \alpha_{2,4} & & \\ \alpha_{1,4} & \alpha_{2,3} & & \\ \alpha_{1,3} & & & \\ \alpha_{1,2}. & & & \end{cases}$$

That is, if $j \geq i$, $i + j \leq m - c - 1$ and $i + j = k + 2$, then $\alpha_{i,j}$ is in the i-th column and in the k-th row of the table (counting from the bottom row to the top row). If $1 < \lambda < \nu$ and $1 + \nu \leq m - c - 1$, we denote by $\mathbf{T}_{\lambda,\nu}$ the subtable of τ_G given by

$$\mathbf{T}_{\lambda,\nu} = \begin{cases} \alpha_{1,\nu} & \cdots & \alpha_{\nu-\lambda+1,\lambda} \\ \vdots & \iddots & \\ \alpha_{1,\lambda}. & & \end{cases}$$

In [8], another useful invariant parameter of a p-group of maximal class is defined by

$$l(G) = l = \min\{i \in \{1, \ldots, (p-1)/2\} \mid x'_i \neq 0\}. \tag{2}$$

If $l = \frac{p-1}{2}$, then $m = c + p$ (cf. [8]).

If $l + c_0 \leq \frac{p-1}{2}$, we define

$$t(G) = t = \min\{i \in \mathbb{N} \mid \alpha'_{i,2l+c_0+1} \neq 0\}. \tag{3}$$

In Lemma 2.1, we will show that t is well-defined.

Throughout this paper, we always consider G as a p-group of maximal class of order p^m, degree of commutativity $c(G) = c$ and invariant parameters $l(G) = l \leq \frac{p-3}{2}$ and $c_0(G) = c_0$. We will use it without more comment and we will omit this fact in the statements of the results.

We define

$$x_i = \frac{x'_i}{x'_l}, \quad \text{if } 2i + 1 \leq m - c - 1; \tag{4}$$

$$y'_j = \alpha_{1,2l+j}, \quad \text{if } 2l + j + 1 \leq m - c - 1; \tag{5}$$

$$y_j = \frac{y'_j}{x'_l}, \quad \text{if } 2l + j + 1 \leq m - c - 1. \tag{6}$$

$$w'_i = \alpha_{i,2l+c_0+1}, \quad \text{if } 2l + c_0 + i + 1 \leq m - c - 1; \tag{7}$$

$$w_i = \frac{w'_i}{w'_t}, \quad \text{if } \max\{2l + c_0 + i + 1, 2l + c_0 + t + 1\} \leq m - c - 1; \tag{8}$$

$$g(r, i) = \frac{f(i, i+1, 2l + t + r - 2i - 1)}{(-1)^t x'_l w'_t}, \quad \text{if } 2l + t + r \leq m - 2c - 1; \tag{9}$$

$$\backslash a, b\backslash = \begin{cases} a(a-1)\cdots(a-b+1), & \text{if } b > 0; \\ 1, & \text{if } b = 0; \quad \forall b \in \mathbb{Z}. \\ \frac{1}{(a-b)(a-b-1)\cdots(a+1)}, & \text{if } b < 0. \end{cases}$$

It is interesting to obtain lower bounds for c because they can be translated into structural properties of G. For example, if we search for the defining relations of G, an improvement of only one unit in the lower bound for c allows us to eliminate many variables in the commutator structure of the defining relations.

Blackburn (cf. [1]) showed that $c = m-2$ if G is a 2-group of maximal class, whereas $c \geq m-4$ for $p = 3$ and $2c \geq m-6$ for $p = 5$. Later, these results were independently generalized to arbitrary p-groups of maximal class by Shepherd (cf. [7]) and Leedham-Green and McKay (cf. [4]), who proved that $2c \geq m - 3p + 6$. However, examples constructed by Leedham-Green and McKay (cf. [5]) suggested that the given lower bound for c could be improved. In fact, Fernández-Alcober showed that $2c \geq m - 2p + 5$ for $p \geq 7$ (cf. [2]). Moreover, this last bound in terms of the order of G is exact, because there exist examples of p-groups of maximal class satisfying $2c = m - 2p + 5$ (cf. [5]).

However, calculations that we have done for p-groups of maximal class with $p \leq 47$ show that if $p \leq 47$, there exists a function $\psi(p, l, c_0)$ such that $2c \geq m - \psi(p, l, c_0)$ and $\psi(p, l, c_0) \leq 2p - 5$. Moreover, we notice that if $p \leq 47$, $\psi(p, l, c_0) < 2p - 5$ for almost all pairs (l, c_0). Thus, the following problem arises: find the expression of $\psi(p, l, c_0)$ for G any p-group of maximal class and p an arbitrary prime. The papers [3,9–11] deal with the search for this function. For G a p-group of maximal class with Y_1 of class 2, Jaikin-Zapirain and Vera-López have found the value $\psi(p, l, c_0)$ in [3]. Moreover, they have given examples of p-groups of maximal class with Y_1 of class 2 satisfying $2c = m - \psi(p, l, c_0)$. But if G is a p-group of maximal class without any restriction on the class of Y_1, the problem is still open.

For each p prime number, we define the following sets:

$$\mathbf{U}_p = \left\{1, 2, \ldots, \frac{p-3}{2}\right\} \times \{0, 1, \ldots, p-2\},$$
$$\mathbf{A}_p = \left\{(l, c_0) \in \mathbf{U}_p \mid l + c_0 = \frac{p-1}{2},\ l \geq \max\left\{\frac{p+5}{6}, 3\right\}\right\},$$
$$\mathbf{b}_{p,1} = \{(l, c_0) \in \mathbf{U}_p \mid p - l \leq c_0 \leq p - 2,\ l \geq 2\},$$
$$\mathbf{b}_{p,2} = \{(l, c_0) \in \mathbf{U}_p \mid c_0 \geq 2p - 4l + 1\},$$
$$\mathbf{b}_{p,3} = \left\{(l, c_0) \in \mathbf{U}_p \mid 2p - 4l - 1 \leq c_0 \leq 2p - 4l + 2,\ l > \frac{p-1}{3}\right\},$$
$$\mathbf{b}_{p,4} = \left\{(l, c_0) \in \mathbf{U}_p \mid c_0 \in \{2p - 4l - 2, 2p - 4l - 4\},\ l \geq \frac{p+1}{3},\ l + c_0 > \frac{p+1}{2}\right\},$$
$$\mathbf{b}_{p,5} = \left\{(l, c_0) \in \mathbf{U}_p \mid l \geq \frac{p+5}{6},\ l + c_0 = \frac{p+1}{2}\right\},$$
$$\mathbf{B}_p = \mathbf{b}_{p,1} \cup \mathbf{b}_{p,2} \cup \mathbf{b}_{p,3} \cup \mathbf{b}_{p,4} \cup \mathbf{b}_{p,5},$$
$$\mathbf{C}_p = \{(l, c_0) \in \mathbf{U}_p \mid l + c_0 = p - 1,\ l < p/3\},$$
$$\mathbf{D}_p = \left\{(l, c_0) \in \mathbf{U}_p \mid l = 1,\ \frac{4p-5}{5} \leq c_0 \leq p - 3\right\},$$
$$\mathbf{e}_{p,1} = \left\{(l, c_0) \in \mathbf{U}_p \mid l \geq 3,\ l + c_0 \leq \frac{p-3}{2},\ 0 \leq c_0 \leq 2l - 1\right\},$$
$$\mathbf{E}_p = \mathbf{e}_{p,1} \cup \{(2, 0)\},$$
$$\mathbf{F}_p = \left\{(l, c_0) \in \mathbf{U}_p - (\mathbf{B}_p \cup \mathbf{C}_p) \mid c_0 = p - 4l + 3\lambda + \mu,\ \lambda \geq 1,\ \mu = 1, 3, 5,\ \frac{p+1}{2} \leq l + c_0\right\},$$
$$\mathbf{g}_{p,1} = \left\{(l, c_0) \in \mathbf{U}_p \mid l + c_0 = \frac{p-1}{2},\ 3 \leq l < \frac{p+5}{6}\right\},$$
$$\mathbf{g}_{p,2} = \left\{(l, c_0) \in \mathbf{U}_p \mid l + c_0 \leq \frac{p-3}{2},\ c_0 > 2l - 1,\ l \geq 3\right\},$$
$$\mathbf{G}_p = \mathbf{g}_{p,1} \cup \mathbf{g}_{p,2},$$
$$\mathbf{J}_p = \mathbf{A}_p \cup \mathbf{B}_p \cup \mathbf{C}_p \cup \mathbf{D}_p \cup \mathbf{E}_p \cup \mathbf{F}_p.$$

Collecting the information obtained in [9–11], Theorem 1 has been proved:

Theorem 1. *Let G be a p-group of maximal class of order p^m, degree of commutativity c and invariant parameters l and c_0. Then, $2c \geq m - \psi(p, l, c_0)$, where $\psi(p, l, c_0)$ is given by*

$$\psi(p, l, c_0) = \begin{cases} 2l + c_0 + 1, & \text{if } (l, c_0) \in \mathbf{A}_p \text{ and } c_0 < c; \\ p + 2l - c_0, & \text{if } (l, c_0) \in \mathbf{B}_p; \\ p + 1, & \text{if } (l, c_0) \in \mathbf{C}_p; \\ 2c_0 - p + 7, & \text{if } (l, c_0) \in \mathbf{D}_p \text{ and } c_0 < c; \\ 2l + c_0 + 2, & \text{if } (l, c_0) \in \mathbf{E}_p; \\ p + 2l - c_0 + 1, & \text{if } (l, c_0) \in \mathbf{F}_p. \end{cases}$$

Besides, there exist p-groups of maximal class of order p^m such that $2c = m - \psi(p, l, c_0)$ for the regions $\mathbf{A}_p$, $\mathbf{B}_p$ and $\mathbf{E}_p$.

Remarks. 1. The condition $c_0 < c$ in $\mathbf{A}_p \cup \mathbf{D}_p$ is not restrictive. Otherwise, $c = c_0$ is given by the defining conditions of $\mathbf{A}_p$ or $\mathbf{D}_p$.

2. Fixing a prime p, we notice that $\mathbf{J}_p$ does not cover all the possible $(l, c_0) \in \mathbf{U}_p$. For example, for $p = 37$, Fig. 1 shows:

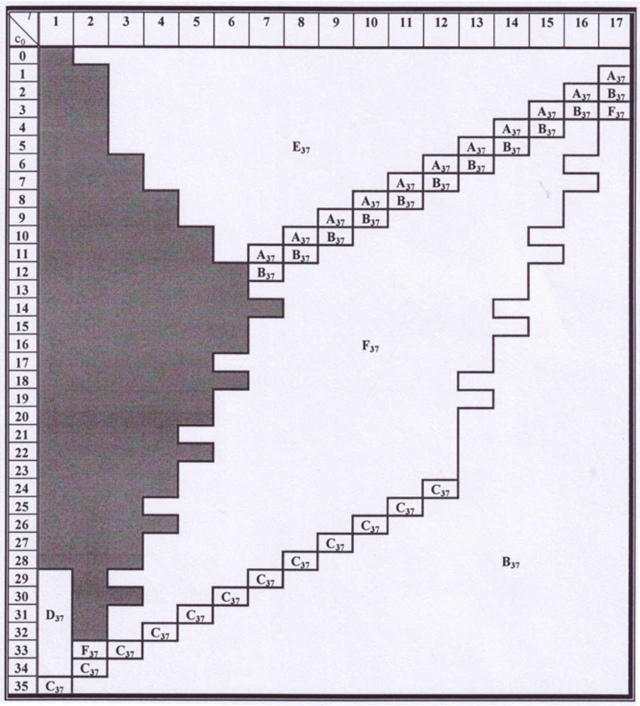

Fig. 1. Regions defined in Theorem 1 for $p = 37$.

(a) The regions $\mathbf{A}_{37}, \mathbf{B}_{37}, \mathbf{C}_{37}, \mathbf{D}_{37}, \mathbf{E}_{37}$ and $\mathbf{F}_{37}$.
(b) The cells corresponding to $(l, c_0) \in \mathbf{U}_{37} - \mathbf{J}_{37}$ in dark grey in order to indicate that $\psi(37, l, c_0)$ has not been defined for these pairs (l, c_0) in Theorem 1.

In this paper, we find $\psi(p, l, c_0)$ for some $(l, c_0) \in \mathbf{U}_p - \mathbf{J}_p$. In fact, we prove the following theorem:

Main Theorem. *Let G be a p-group of maximal class of order p^m, degree of commutativity c and invariant parameters $l \geq 3$ and c_0 satisfying $l + c_0 \leq \frac{p-1}{2}$. Then, $2c \geq m - (2l + c_0 + 3)$ holds.*

It is easy to check that the region $\{(l, c_0) \in \mathbf{U}_p \mid l + c_0 \leq \frac{p-1}{2}, l \geq 3\}$ that appears in the statement of the Main Theorem is $\mathbf{A}_p \cup \mathbf{e}_{p,1} \cup \mathbf{G}_p$. We obtain a new lower bound for $c(G)$ in the region $\mathbf{G}_p$ that improves on the best known lower bound for c in terms of the order of G (cf. [2]). That is, we prove that $\psi(p, l, c_0) = 2l + c_0 + 3$ if $(l, c_0) \in \mathbf{G}_p$. The number of pairs $(l, c_0) \in \mathbf{U}_p - \mathbf{J}_p$ for which we have defined $\psi(p, l, c_0)$ in the Main Theorem is

$$\begin{aligned} |\mathbf{G}_p| &= |\mathbf{g}_{p,1}| + |\mathbf{g}_{p,2}| \\ &= \left[\frac{p+5}{6}\right] + \epsilon + \frac{\left(p - 10 - 3\left[\frac{p-3}{6}\right]\right)\left(\left[\frac{p-3}{6}\right] - 2\right)}{2}, \end{aligned}$$

where ϵ is equal to -2 if $\frac{p+5}{6} \notin \mathbb{N}$, or -3, otherwise. For example, for the prime $p = 37$, the regions $\mathbf{J}_{37}$ and $\mathbf{G}_{37}$ appear in Fig. 2 and $|\mathbf{G}_{37}| = 22$.

Besides, there exist examples of p-groups of maximal class with invariant parameters l and c_0 such that $l + c_0 \leq \frac{p-3}{2}$ and $2c = m - (2l + c_0 + 2)$ and examples of p-groups of maximal class with invariant parameters l and c_0 such that $l + c_0 \leq \frac{p-1}{2}$ and $2c = m - (2l + c_0 + 1)$ (cf. Theorem 5.5 of [3]). So, the bound given in the Main Theorem is almost exact.

The proof of the Main Theorem is based on the following facts:

1. Arguing by contradiction, we assume that $2l + c_0 + 3 \leq m - 2c - 1$ if $l \geq 3$ and $l + c_0 \leq \frac{p-1}{2}$.
2. By Lemma 2.1, we know that $t \leq c_0 + 1$ if $l + c_0 < \frac{p+1}{2}$. So, if we deduce that $t > c_0 + 1$, we derive a contradiction and the Main Theorem is proved.
3. In order to show that $t > c_0 + 1$, we study separately the cases $t \leq c_0 - 3$, and $t = c_0 - i$, with $i = -1, 0, 1, 2$. First, we determine the values of $\alpha_{i,j}$ in the triangles $\mathbf{T}_{2l+c_0+1,t+2l+c_0}$ and $\mathbf{T}_{t,t+2l+j}$ for some j. We will manage to bring these two triangles close enough for the known values of τ_G to allow us to derive a contradiction, by using (P1)–(P5).
4. Lemmas and propositions of Sections 2–6 are used to determine the values of some particular $\alpha_{i,j}$. Conditions of type $2l + t + k \leq m - 2c - 1$ appear in the statements of these results. Obviously, we choose k satisfying $t + k \leq c_0 + 3$.

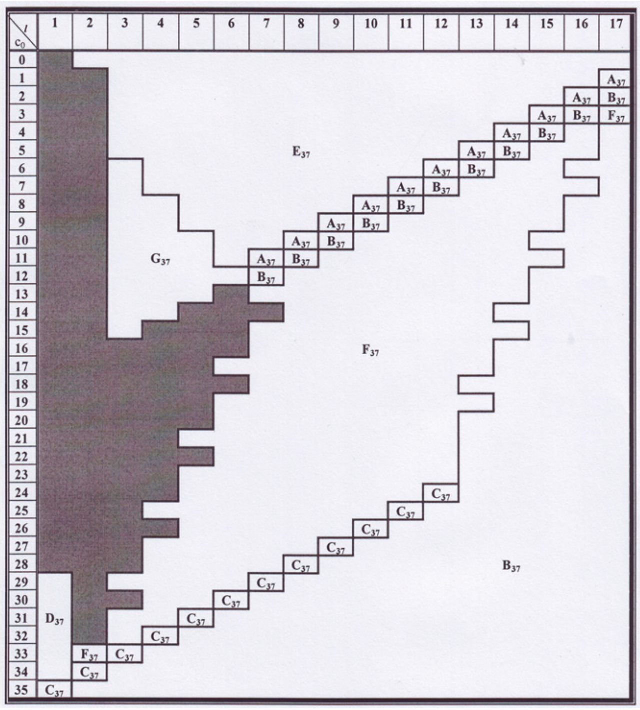

Fig. 2. Regions defined in Theorem 1 and the Main Theorem for $p = 37$.

The scheme of the paper is the following one:

In Section 2, we check that t, defined in (3), is well-defined. We also prove the Main Theorem in the particular case of $l > c_0 + 2$. Moreover, we obtain relations between the invariant parameters of G, the p-group of maximal class, and $H \leq G$, the maximal subgroup of G, that are useful for applying inductive arguments. In addition, we compute the determinant of a matrix that will appear in Section 7. We know that $w_1 = \cdots = w_{t-1} = 0$ and $w_t = 1$. We are interested in computing w_{t+k} for $k \geq 1$. Hence, we also deduce that under a certain hypothesis, $w_{t+2} = 0$ or $w_{t+1} \equiv (2t - 2l + 3) \bmod p$.

In Section 3, we conclude $w_{t+1} \not\equiv (2t - 2l + 3) \bmod p$ and $w_{t+2} = 0$ if some conditions are satisfied.

In Section 4, we look for the value of w_{t+1}. Lemmas of this section show that $w_{t+1} = 0$ or $w_{t+1} = -1$ if certain conditions are satisfied.

In Sections 5 and 6, we analyze the cases $w_{t+1} = -1$ and $w_{t+1} = 0$, respectively, and we obtain the values of $y_{t+\delta}$ for $-2l + 2 \leq \delta \leq k - 2$ if $2l + t + k \leq m - 2c - 1$ and other additional hypotheses hold.

As we have said, all the information of Sections 2–6 is used to prove the Main Theorem in Section 7.

Finally, we define

$$\mathbb{M}(G) = \{H \leq G \mid H \text{ is maximal in } G \text{ and } H \neq Y_1(G), C_G(Y_{m-2})\}.$$

We know that if G is a p-group of maximal class, then $H \in \mathbb{M}(G)$ is a p-group of maximal class with $c(H) = c(G) + 1$ (cf. [12]). Henceforth, in inductive arguments, we use the following notation: $c(G)$, $c(H)$, $l(G)$, $l(H)$, $\alpha_{i,j}(G)$ and $\alpha_{i,j}(H)$ depending on the p-group of maximal class G or the maximal subgroup $H \leq G$, which we work on. If the group does not appear, we understand that we are working on G and we write c, l or $\alpha_{i,j}$ and so on, instead of $c(G)$, $l(G)$ or $\alpha_{i,j}(G)$ and so on, respectively.

2. Preliminary lemmas

In this section, we show that t, defined in (3), is well-defined. This invariant parameter is necessary for proving the Main Theorem. Moreover, we prove some general interesting results that we will use in the following sections.

Lemma 2.1. *Let G be a p-group of maximal class with invariant parameters l and c_0. If $w'_i = 0$ for $i = 1, \ldots, c_0 + 1$, then $l + c_0 \geq \frac{p+1}{2}$.*

Proof. We know that $x'_1 = \cdots = x'_{l-1} = 0$ and $x'_l \neq 0$. From (1), the system

$$\left\{ w'_i = 0, \quad \text{for } i = 1, \ldots, c_0 + 1 \right\}$$

is a homogeneous linear system of c_0+1 equations in the c_0+1 variables $x'_l, \ldots, x'_{l+c_0}$. As $x'_l \neq 0$, the determinant of the system matrix, $\det(A)$, is 0 and from Lemma 1 of [9], it follows that

$$\det(A) = \frac{F_1(l+c_0+1, c_0+2, c_0+1)}{F_1(c_0+2, c_0+2, c_0+1)},$$

where

$$F_1(r,s,u) = \prod_{1\le k\le s-1} (r-k)^{\min\{k,s-u,s-k\}} \cdot \prod_{s-u+1\le k\le s+u-3} (2r-k-1)^{\min\left\{\left[\frac{u-s+k+1}{2}\right],\left[\frac{u+s-k-1}{2}\right]\right\}}. \tag{10}$$

This implies that $l+c_0 \ge \frac{p+1}{2}$. $\square$

From Lemma 2.1, we notice that if $l+c_0 \le \frac{p-1}{2}$, then there exists $w'_i \neq 0$ for some $1 \le i \le c_0+1$. So, the invariant parameter t, given in (3), is well-defined.

Lemma 2.2. *If G is a p-group of maximal class with invariant parameters l and c_0 such that $l+c_0 \le \frac{p-1}{2}$ and $l > c_0+2$, then $2c \ge m-(2l+c_0+3)$.*

Proof. Arguing by contradiction, we assume that $2l+c_0+3 \le m-2c-1$. Then, $0 = f(j,l,l+1) = -x'_l w'_j$ for $j < c_0+2$. But, this implies that $w'_j = 0$ for $j = 1, \ldots, c_0+1$. By applying Lemma 2.1, it follows that $l+c_0 \ge \frac{p+1}{2}$, a contradiction. $\square$

Henceforth, we assume that

$$l+c_0 \le \frac{p-1}{2} \quad \text{and} \quad l \le c_0+2. \tag{11}$$

It is easy to check that if $2l+c_0+3 \le m-2c-1$ and (11) hold, then $f(j,l,l+1) = 0$, with $j < l$, implies that $t \ge l$.

Lemma 2.3. *If $y'_{s+i} = 0$ for $i = 0, 1, \ldots, s-1$, where $s \ge 0$, then*

$$l+s-j \equiv 0 \bmod p, \quad \textit{for some } j \in \{1,\ldots,s\},$$

or

$$2l+2s-\omega-1 \equiv 0 \bmod p, \quad \textit{for some } \omega \in \{2,\ldots,2s-2\}.$$

Proof. If $y'_{s+i} = 0$ for $i = 0, 1, \ldots, s-1$, then (P3) implies that $\alpha_{i,2l+s} = 0$ for $i = 1, \ldots, s$. According to (1) and $0 = x'_k$, $k = 1, \ldots, l-1$, the homogeneous linear system

$$\left\{ 0 = \alpha_{i,2l+s} = \sum_{k=i}^{\left[\frac{2l+s+i-1}{2}\right]} (-1)^{k-i} \binom{2l+s-k-1}{k-i} x'_k,\ i = 1, \ldots, s, \right\}$$

in variables x'_k has a non-zero solution. From Lemma 1 of [9], the determinant of the system matrix $\det(A)$ is

$$\det(A) = \pm\frac{F_1(l+s, s+1, s)}{F_1(s+1, s+1, s)},$$

where F_1 is given in (10). This completes the proof. $\square$

It is helpful to obtain relations between the invariant parameters of G, the p-group of maximal class, and $H \in \mathbb{M}(G)$, the maximal subgroup of G. It is easy to check:

Lemma 2.4. *Let G be a p-group of maximal class and $H \in \mathbb{M}(G)$. Then,*

(i) $m(H) = m(G) - 1$.
(ii) $l(H) = l(G) - 1$.
(iii) $c_0(H) = c_0(G) + 1$ *if* $c_0(G) \le p-2$.
(iv) $t(H) = t(G) - 1$.
(v) $\alpha_{i,j}(H) = \alpha_{i+1,j+1}(G)$ *if* $i+j \le m(G) - c(G) - 3$.
(vi) $w_i(H) = w_{i+1}(G)$. *In particular,* $w_{t(H)+j}(H) = w_{t(G)+j}(G)$ *for all* $j \ge 0$ *such that* $2l(G)+c_0(G)+t(G)+j+1 \le m-c-1$.

Remarks. 1. We notice that if G is a p-group of maximal class and $H \in \mathbb{M}(G)$, then

$$m(G) - 2c(G) - (2l(G)+t(G)) = m(H) - 2c(H) - (2l(H)+t(H)).$$

Hence, conditions of type $2l+t+k \le m-2c-1$ are good for applying the inductive argument on G. However if we consider the hypothesis $2l+c_0+3 \le m-2c-1$, then $2l(G)+c_0(G)+3 \le m(G)-2c(G)-1$ does not imply $2l(H)+c_0(H)+3 \le m(H)-2c(H)-1$ and we cannot use the inductive argument on G.

2. Bearing in mind the relation between $\alpha_{i,j}(H)$ and $\alpha_{i+1,j+1}(G)$ given in Lemma 2.4(v), it follows that $y'_j(H) = \alpha_{2,2l(G)+j-1}(G)$. Hence, we do not use only inductive arguments on G when we work with the y_j's.

The following lemma gives us the interesting particular cases to study:

Lemma 2.5. *If $p > 2$, $l \geq 3$ and $2l + t + 3 \leq m - 2c - 1$, then one of the following assertions holds:*

(i) $w_{t+2} = 0$.
(ii) $2w_{t+1} \equiv 2t - 2l + 3 \bmod p$.

Proof. We argue by induction on $l + |G|$. If $l = 3$, we define

$$\beta_1(i) = \begin{cases} (2w_{t+2} - 2tw_{t+1} + t^2 - t)(t - 1 - w_{t+1}), & \text{if } i = 1; \\ 2((t-2)^2 - (2t-3)w_{t+1} + 2w_{t+2})(2w_{t+1} - 2t + 3), & \text{if } i = 2; \end{cases}$$

$$\beta_2(i) = \begin{cases} (2tw_{t+1} - 2w_{t+2} - t^2 + t), & \text{if } i = 1; \\ 2(t-2)(2t-3) + 2(5-4t)w_{t+1} + 8w_{t+2}, & \text{if } i = 2; \end{cases}$$

$$\beta_3(i) = \begin{cases} 2t - 2w_{t+1}, & \text{if } i = 2; \\ 8w_{t+1} - 8t + 12, & \text{if } i = 3. \end{cases}$$

Straightforward calculations show that

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \beta_i(j) g(i,j) + \sum_{k=2}^{3} \beta_3(k) g(3,k) = (-1)^{t-1} 4(2w_{t+1} - 2t + 3) w_{t+2}. \tag{12}$$

Then, bearing in mind that $g(r, i) = 0$ for $1 \leq r, i \leq 3$, the equality (12) implies that the result is true for $l = 3$. For $l > 3$, it is enough to apply an inductive argument and Lemma 2.4. □

Furthermore, in order to prove that $t \neq c_0 + 1$ when $l > 3$ and $l + c_0 \leq \frac{p-1}{2}$, we need to compute determinants of some matrices whose entries are combinatorial numbers. In this computation, we use the following equality of combinatorial numbers:

Lemma 2.6. *Let $0 \leq k \leq n$, $M \leq n$ and $n - M \leq 2k$. Then,*

$$\sum_{j=0}^{M} \binom{M}{j} \binom{n}{k+j} \binom{2k+2j}{n+M} = \binom{n}{k} \binom{2k}{n-M}.$$

Proof. In order to prove the equality

$$\sum_{j=0}^{M} \binom{M}{j} \binom{n}{k+j} \binom{2k+2j}{n+M} = \binom{n}{k} \binom{2k}{n-M},$$

it is enough to check that

$$\sum_{j=0}^{M} \binom{M}{j} \backslash k+M, M-j \backslash\backslash n-k, j \backslash\backslash 2k-n+M, 2M-2j \backslash\backslash 2k+2j, 2j \backslash = \backslash k+M, M \backslash\backslash n+M, 2M \backslash.$$

Let $E(n, k, M)$ be the left-hand side of the previous equality. Then, it is easy to prove that:

(a) $E(n, k, M)$ is a polynomial in variables k, n with degree up to $3M$.
(b) $\backslash k+M, M\backslash$ is a factor of $E(n, k, M)$.
(c) $\backslash n+M, 2M\backslash$ is a factor of $E(n, k, M)$.

Therefore, $E(n, k, M) = \lambda \backslash k+M, M \backslash\backslash n+M, 2M \backslash$ for some λ. But if $k = 0$, $n = M$, then $E(M, 0, M) = \backslash M, M \backslash\backslash 2M, 2M \backslash$. Hence, $\lambda = 1$ and

$$E(n, k, M) = \backslash k+M, M \backslash\backslash n+M, 2M \backslash. \quad \square$$

Remark. The last identity can be verified by using Zeilberger's Algorithm explained in [6].

Now, we compute the determinant of a matrix that appears in Section 7:

Lemma 2.7. *Let $A \in \mathrm{Mat}_{n\times n}(K[x]) = (a_{i,j})$ be defined by*

$$a_{ij} = \begin{cases} \binom{x-j}{n+1+i-2j} & \text{if } 1 \leq i \leq n-1,\ 1 \leq j \leq n; \\ \binom{x-\lambda-j}{n+2-\lambda-2j} & \text{if } i = n,\ 1 \leq j \leq n;\ 1 \leq \lambda \leq n. \end{cases}$$

Then,

$$\det(A) = (-1)^{n-1} \cdot 2^{1-[\frac{\lambda+1}{2}]} \cdot \prod_{j=1}^{n-1} \frac{j!}{(2n-2j)!} \cdot \frac{\backslash n-1, \lambda-1 \backslash \cdot \backslash n+\lambda-2, \lambda-1 \backslash}{\backslash 2x-n-2, \lambda-1 \backslash} \cdot \prod_{w=4}^{2n} (2x-w)^{\min([\frac{w-2}{2}],[\frac{2n-w+2}{2}])}.$$

Proof. We make the following row and column transformations in A:

$$\left\{\begin{array}{ll} A^j \longrightarrow \frac{(2n+1-2j)!}{\backslash x-j,n+3-2j-1\backslash}A^j, & j=1,\ldots,n-1,\\ A_i \longrightarrow A_i+2A_{i+1}, & \\ A_i \longrightarrow \frac{1}{2x-2-i-s}A_i, & i=1,\ldots,s,\ s=n-2,\ldots,1,\\ A^j \longrightarrow \frac{1}{2n-2j+1}A^j, & j=1,\ldots,n-1,\\ A^n \longrightarrow \backslash x-2,n-2\backslash A^n, & \end{array}\right\}$$

where A^j is the j-th column of A and A_i is the i-th row of A.

Then, we obtain $C=(c_{ij})$ defined by

$$c_{ij}=\begin{cases}\backslash x-(n+2-j),i-1\backslash, & \text{if } 1\le i\le n-1;\\ \frac{\backslash 2n-2j,n+\lambda-2\backslash}{\backslash x-j,\lambda\backslash}, & \text{if } i=n;\end{cases}$$

and

$$\det(A)=2^{\frac{n^2-2n+\delta}{4}}\prod_{j=1}^{n-1}\frac{1}{(2n-2j)!}\cdot\prod_{w=1}^{n}(x-w)^{\min(w,n+1-w)}\cdot\prod_{\substack{w=5\\ w\text{ odd}}}^{2n-3}(2x-w)^{\min(\frac{w-3}{2},\frac{2n-w-1}{2})}\cdot\det(C).$$

In order to compute $\det(C)$, we use the cofactor expansion along the last row of C. The (n,i)-cofactor of C is $(-1)^{n-i}\binom{n-1}{i-1}P(n)$, where

$$P(n)=0!\cdot 1!\cdot 2!\cdots(n-2)!\,.$$

Indeed if we consider the matrix R obtained by substitution of the n-th row of C by $(1,y,y^2,\ldots,y^{n-1})$, then $\det(R)$ is a polynomial in variables x and y, that is, $\det(R)=R(x,y)$. However, $\det(C)$ does not depend on x. In fact, the transformations

$$R_i\longrightarrow iR_i+R_{i+1},\quad i=n-2,\ldots,1,$$

yield $R(x,y)$ to $R(x+1,y)$, so $R(x,y)$ is a polynomial in x invariant under translations. Therefore, $\det(R)$ does not depend on x. Moreover $\det(R(x,y))=0!\cdot 1!\cdot 2!\cdots(n-2)!\cdot(y-1)^{n-1}$. Hence,

$$\det(C)=\sum_{j=1}^{n}(-1)^{n-j}\,P(n)\binom{n-1}{j-1}\cdot c_{nj}=P(n)g_n(x),$$

with

$$g_n(x)=\sum_{j=1}^{n}(-1)^{n-j}\,\frac{a_j}{\backslash x-j,\lambda\backslash}$$

and

$$a_j=\binom{n-1}{j-1}\cdot\binom{2n-2j}{n+\lambda-2}\cdot(n+\lambda-2)!.$$

But, we claim that $g_n(x)=f_n(x)$, where

$$f_n(x)=(-1)^{n-1}\binom{n-1}{\lambda-1}(n+\lambda-2)!\cdot\frac{\backslash 2x-(n+\lambda+1),n-\lambda\backslash}{\backslash x-1,n\backslash}.$$

Indeed, as the degree of the numerator is lower than the degree of the denominator, it is enough to check that the two functions have the same residue at the points $x=k$ for $1\le k\le n$. Let $r_k=[(x-k)f_n(x)]_{x=k}$. Then,

$$\begin{aligned} r_k &= (-1)^{n-1+n-k}\binom{n-1}{\lambda-1}(n+\lambda-2)!\,\frac{\backslash 2k-(n+\lambda+1),n-\lambda\backslash}{(k-1)!(n-k)!}\\ &=\begin{cases}0, & \text{if } n\le 2k-\lambda-1;\\ (-1)^{n-1-\lambda-k}\binom{n-1}{\lambda-1}(n+\lambda-2)!\,\frac{\backslash 2n-2k,n-\lambda\backslash}{(k-1)!(n-k)!}, & \text{if } n\ge 2k-\lambda.\end{cases}\end{aligned}$$

Let $u_k=[(x-k)\,g_n(x)]_{x=k}$. Then,

$$u_k=\sum_{j=\max(1,k-\lambda+1)}^{k}(-1)^{n-j+\lambda-k+j-1}\frac{\binom{n-1}{j-1}\binom{2n-2j}{n+\lambda-2}(n+\lambda-2)!}{(k-j)!(\lambda-k+j-1)!}.$$

If $n \leq 2k - \lambda - 1$, all terms of u_k are 0 because the factor $\binom{2n-2j}{n+\lambda-2}$ of each term is 0. If we put

$$\begin{cases} \lambda = M + 1, & n = n' + 1, \\ k = n' + 1 - k'', & j = n' + 1 - k'' - j'', \end{cases}$$

we have

$$u_k = (-1)^{M-k''} \frac{(n'+M)!}{M!} \sum_{j''=0}^{M} \binom{M}{j''} \binom{n'}{k''+j''} \binom{2k''+2j''}{n'+M}.$$

Now, by Lemma 2.6, we conclude that $u_k = r_k$ for each $k \leq \frac{n+\lambda}{2}$. □

3. The case $2w_{t+1} \equiv 2t - 2l + 3 \bmod p$

We recall that if $l + c_0 \leq \frac{p-1}{2}$, then $w_i = 0$ for $1 \leq i \leq t-1$ and $w_t = 1$. Furthermore for $l \geq 3$ and $2l+t+3 \leq m-2c-1$ holding, we have shown that $w_{t+2} = 0$ or $2w_{t+1} \equiv (2t - 2l + 3) \bmod p$. Now, we obtain the values of y_{t-i} for some i, for when $2l + t + 2 \leq m - 2c - 1$ and $2w_{t+1} \equiv (2t - 2l + 3) \bmod p$. Moreover, we prove that if $2l + t + 6 \leq m - 2c - 1$ holds, then $2w_{t+1} \not\equiv (2t - 2l + 3) \bmod p$.

Lemma 3.1. *Suppose that $l = 3$, $2w_{t+1} \equiv (2t - 3) \bmod p$ and $2l + t + 2 \leq m - 2c - 1$. Then,*

(i) $y_{t-i} = 0$ *for* $i = 0, 1, 2$.
(ii) $y_{t-4} = \frac{3}{2} y_{t-3}$.
(iii) $y_{t-5} = \frac{(-1)^{t-1}}{2} t + \frac{3}{4}(-1)^t + \frac{15}{8} y_{t-3}$.
(iv) $y_{t-6} = \frac{19(-1)^t}{8} - \frac{5}{4}(-1)^t t + \frac{35}{16} y_{t-3}$.

Proof. Immediate. It is the solution of the homogeneous linear system $g(r, i) = 0$, with $1 \leq r \leq 2$ and $1 \leq i \leq 3$ written in variables y_{t-i}. □

By using inductive argument, this last lemma can be generalized for $l \geq 3$ as follows:

Proposition 3.2. *If $l \geq 3$, $2w_{t+1} \equiv (2t - 2l + 3) \bmod p$ and $2l + t + 2 \leq m - 2c - 1$, then $y_{t-i} = 0$ for $i = 0, \ldots, 2l - 4$ and $y_{t-2l+2} = \frac{2l-3}{2} y_{t-2l+3}$.*

Proof. We argue by induction on $l + |G|$. From Lemma 3.1(i) and (ii), the result is true for $l = 3$. For $l > 3$, working with $H \in \mathbb{M}(G)$, and bearing in mind (P3), $g(1, 1) = 0$ and $g(1, l-1) = 0$, an inductive argument completes the proof. □

For $l = 3$ and $2l + t + 5 \leq m - 2c - 1$, we obtain the values of some w_{t+i} and y_{t-j} in the following lemma:

Lemma 3.3. *If $l = 3$, $p > 7$, $2l+t+5 \leq m-2c-1$, $(2t-1)(2t-3)(2t-5)(2t-7)(2t-9)(2t-15)(t-6)(t+6)t(t+3)(t-1) \not\equiv 0 \bmod p$ and $2w_{t+1} \equiv (2t - 3) \bmod p$, then*

$$w_{t+2} = \frac{(2t-3)(4t-3)}{18}, \qquad w_{t+3} = \frac{t(2t-3)(4t-3)}{81}, \qquad w_{t+4} = 0,$$

$$y_{t+1} = 0, \qquad y_{t-3} = 2(-1)^t \frac{4t-3}{2t+3},$$

$$y_{t-7} = \frac{(-1)^t}{4} \frac{t(t^3 - 68t^2 + 513t - 846)}{(2t+3)(4t-9)},$$

$$y_{t-8} = \frac{(-1)^t}{8} \frac{7t^4 - 200t^3 + 1287t^2 - 1926t - 648}{(2t+3)(4t-9)},$$

$$y_{t-9} = -\frac{(-1)^t}{24} \frac{-68040 + 3974t^4 - 183t^5 + t^6 - 30801t^3 - 83970t + 95859t^2}{(4t-9)(4t-15)(2t+3)},$$

$$y_{t-10} = -\frac{(-1)^t}{16} \frac{-77760 + 3667t^4 - 215t^5 + 3t^6 - 25053t^3 - 61992t + 73350t^2}{(4t-9)(4t-15)(2t+3)}.$$

Proof. It is easy to check that

$$(9 - 6t) w_{t+3} \equiv (4t^2 - 10t + 6 - 12 w_{t+2}) w_{t+2} \bmod p. \tag{13}$$

Otherwise, we derive a contradiction, by applying Lemma 3.1 and the following two polynomial combinations:

$$\begin{aligned}0 &= -12(w_{t+3}g(3,3) - w_{t+2}g(4,3)) \\ &= y_{t-3}(2t-1)((9-6t)w_{t+3} - (4t^2-10t+6-12w_{t+2})w_{t+2}) \\ 0 &= 2((24t^2+18-48t)w_{t+2} + 39t + 36t^3 - 8t^4 - 9w_{t+3} - 58t^2 - 24w_{t+3}t - 9)g(3,3) \\ &\quad + 3(-8t^3+28t^2+16w_{t+2}t - 30t + 9 + 6w_{t+2})g(4,3) \\ &= 4y_{t-5}(2t-1)((9-6t)w_{t+3} - (4t^2-10t+6-12w_{t+2})w_{t+2}).\end{aligned}$$

In addition, we have the following identity:

$$\begin{aligned}0 &= -\frac{1}{48}(144t^3 - 804t^2 + 1476t - 891 - (64t^2+176t+13)w_{t+2} + 192w_{t+2}^2)g(3,3) \\ &\quad - \frac{1}{6}(-12t^2+36t-27+(8t^2+4t-24)w_{t+2} - 24w_{t+2}^2)g(3,4) + g(4,4)(2t-4-w_{t+2})(2t-3) \\ &= (-1)^{t-1}\frac{1}{96}(2t-1)(2t-3)(2t-5)(2t-7)(8t^2-18t+9-18w_{t+2}). \end{aligned} \tag{14}$$

Therefore, from (13) and (14), it follows that

$$w_{t+2} = \frac{(2t-3)(4t-3)}{18}, \qquad w_{t+3} = \frac{(2t-3)(4t-3)t}{81}.$$

Moreover, for $p > 7$, from $0 = g(3,3)$, we deduce the value of y_{t-3}. Hence, by substituting the known values of y_{t-j} for $j = -1, \ldots, 6$, and w_{t-i} for $i = 0, 1, 2$, we obtain the values of y_{t-k} for $k = 7, 8, 9$ and 10 by solving the following homogeneous linear system:

$$\{0 = g(i,j), i = 1, 2, 3, j = 4, 5\}. \tag{15}$$

Finally, bearing in mind Lemma 3.1 and the value of w_{t+2}, the solution of the homogeneous linear system

$$\begin{Bmatrix} 0 = g(3,1) \\ 0 = g(5,3) \end{Bmatrix}$$

is $y_{t+1} = 0$ and $w_{t+4} = 0$. □

Remarks. 1. By using inductive argument and $t \le c_0 + 1$, it is immediately proved that if $l \ge 3$, $l + c_0 \le \frac{p-1}{2}$ and $2l + t + 5 \le m - 2c - 1$ hold, then $w_{t+4} = 0$.
2. In the proof of Lemma 3.3, when we solve the linear system (15), it is easy to check that $(4t-9)(4t-15) \not\equiv 0 \bmod p$ if $t(t-1)(t+3)(t-6) \not\equiv 0 \bmod p$. We use this fact in the following result.

Lemma 3.4. *If $l = 3$, $p > 7$, $2l + t + 6 \le m - 2c - 1$, $(2t-1)(2t-3)(2t-5)(2t-7)(2t-9)(2t-15)(t-6)(t+6)t(t+3)(t-1)(t+2)(t-3)(t-9)(t-12) \not\equiv 0 \bmod p$ and $2w_{t+1} \equiv (2t-3) \bmod p$, then*

(i) $w_{t+5} = 0$, $y_{t+2} = 0$ *and* $y_{t+3} = 0$.
(ii) $112t^4 - 3384t^3 + 14598t^2 - 9801t - 17010 \equiv 0 \bmod p$.

Proof. By hypothesis, $g(r,i) = 0$ for $1 \le r \le 6$ and $1 \le i \le 6$. Moreover, by Lemma 3.1 and Proposition 3.2, we know the values of y_{t-j} for $j = -1, \ldots, 10$ and w_{t+i} for $i = 2, \ldots, 4$. Then, (i) follows directly from the homogeneous linear system

$$\begin{Bmatrix} g(5,i) = 0, \quad i = 1, 2 \\ g(6,3) = 0. \end{Bmatrix}.$$

Now we conclude (ii) by substituting in $g(6,5) = 0$ the values of y_{t-j} for $j = -3, \ldots, 10$ and w_{t+i} for $i = 1, \ldots 5$ given in Lemmas 3.1, 3.3 and 3.4(i). □

Remark. We know that if $l + c_0 \le \frac{p-1}{2}$, then $t \le c_0 + 1$. So, the condition $(2t-1)(2t-3)(2t-5)(2t-7)(2t-9)(2t-15)(t-6)(t+6)t(t+3)(t-1)(t+2)(t-3)(t-9)(t-12) \not\equiv 0 \bmod p$ holds if we work on a p-group of maximal class such that the invariant parameters l and c_0 satisfy $l + c_0 \le \frac{p-1}{2}$ and $l \ge 3$.

Now, we obtain the value of w_{t+2} and information on w_{t+1} for $l = 3$.

Lemma 3.5. *If $l = 3$, $p > 11$, $2l + t + 6 \le m - 2c - 1$ and $l + c_0 \le \frac{p-1}{2}$, then $2w_{t+1} \not\equiv 2t - 3 \bmod p$ and $w_{t+2} = 0$.*

Proof. Arguing by contradiction, we assume that $2w_{t+1} \equiv 2t - 3 \bmod p$. Then, from Lemma 3.4, it follows that

$$f_1(t) = 112t^4 - 3384t^3 + 14598t^2 - 9801t - 17010 \equiv 0 \bmod p.$$

Moreover, the values of w_{t+i} and y_{t-j} computed in Lemmas 3.1, 3.3 and 3.4 give

$$0 = g(6,4) = \frac{(-1)^t}{810}\,\frac{(2t-1)(2t-3)(14t^2-27t-27)t(4t-3)}{(4t-9)}.$$

So, $g_1(t) = 4t-3 \equiv 0 \bmod p$ or $g_2(t) = 14t^2-27t-27 \equiv 0 \bmod p$. Set $f_1(t) = g_i(t)h_i(t) + r_i(t)$ for $i = 1,2$, where $\deg(r_i(t)) < \deg(g_i(t))$. Then, $r_i(t) \equiv 0 \bmod p$, a contradiction if $p > 11$. Therefore, $2w_{t+1} \not\equiv 2t-3 \bmod p$ and, from Lemma 2.5, $w_{t+2} = 0$. □

By an inductive argument, we extend Lemma 3.5 as follows:

Proposition 3.6. *If $2l+t+6 \le m-2c-1$, $l \ge 3$, $p > 11$, $l + c_0 \le \frac{p-1}{2}$, then $2w_{t+1} \not\equiv 2t-2l+3 \bmod p$ and $w_{t+2} = 0$.*

4. The values of w_{t+1}

In this section, we analyze the case $2w_{t+1} \not\equiv (2t-2l+3) \bmod p$. First of all, by Lemma 2.5, we notice that $w_{t+2} = 0$. In the following results, we prove that, under certain hypotheses, the only possible values of w_{t+1} are 0 or -1.

Lemma 4.1. *If $l = 3$, $2l+t+3 \le m-2c-1$ and $2w_{t+1} \not\equiv (2t-3) \bmod p$, then one of the following assertions holds:*

(i) $w_{t+1} + w_{t+1}^2 \equiv 0 \bmod p$ *and* $w_{t+2} = 0$.
(ii) $w_{t+2} = 0$ *and* $y_{t-4} = y_{t-3} = y_{t-2} = y_{t-1} = y_t = 0$.

Proof. From Lemma 2.5, set $w_{t+2} = 0$. Now, $g(r,i) = 0$ for $r \le 3$ implies

$$\begin{aligned} 0 &= (t-w_{t+1})g(3,2) + \left(tw_{t+1}^2 - w_{t+1}^2 - \frac{3}{2}w_{t+1}t^2 - w_{t+1} + \frac{3}{2}tw_{t+1}r + \frac{1}{2}t^3 + \frac{1}{2}t - t^2\right)g(1,1) \\ &\quad + \frac{1}{2}t(-t+1+2w_{t+1})g(2,1) \\ &= -y_{t-2}(w_{t+1}+w_{t+1}^2). \end{aligned}$$

Therefore, $w_{t+1} + w_{t+1}^2 \equiv 0 \bmod p$ or $y_{t-2} = 0$. If $y_{t-2} = 0$, then the equality $0 = g(1,1) = -y_{t-1} + 2y_{t-2} = -y_{t-1}$ implies $y_{t-1} = 0$. Bearing in mind

$$\begin{aligned} &2w_{t+1} \not\equiv (2t-3) \bmod p, \\ &y_{t-1} = y_{t-2} = 0, \\ &0 = g(2,2), \end{aligned}$$

we obtain $y_{t-3} = 0$. Moreover, $0 = g(1,2)$ implies $y_{t-4} = 0$.

Finally, assume that $y_t \ne 0$. Then, from $0 = g(2,1)$, we deduce $w_{t+1} = t$. Now $0 = g(3,2) = \frac{1}{2}(t+1)ty_t$, a contradiction. □

The last lemma is extended to $l > 3$ as follows:

Lemma 4.2. *If $l \ge 3$, $2l+t+3 \le m-2c-1$ and $2w_{t+1} \not\equiv (2t-2l+3) \bmod p$, then one of the following conditions is satisfied:*

(i) $w_{t+1} + w_{t+1}^2 = 0$ *and* $w_{t+2} = 0$.
(ii) $y_{t-i} = 0$ *for* $i = 0, \ldots, 2l-2$ *and* $w_{t+2} = 0$.

Proof. It is enough to apply Lemma 4.1, induction on $l + |G|$ and (P3). □

Lemma 4.3. *If $l = 3$, $2l+t+6 \le m-2c-1$, $2w_{t+1} \not\equiv (2t-3) \bmod p$ and*

$$t(t+1)(t+2)(t+3)(t-2)(t-3)(2t-3)(2t-5)(2t-7)(2t-9) \not\equiv 0 \pmod p,$$

then $w_{t+1} \in \{0, -1\}$.

Proof. Arguing by contradiction, we assume $w_{t+1} + w_{t+1}^2 \not\equiv 0 \bmod p$. From Lemma 4.1, we deduce $w_{t+2} = 0$, $y_{t-i} = 0$ for $i = 0, 1, \ldots, 4$. Moreover, $g(1,3) = 0 = g(4,3)$ implies

$$y_{t-6} = \frac{(-1)^t + 5y_{t-5}}{2}$$

and $w_{t+3} = 0$.

The system

$$\left\{\begin{matrix} 0 = g(3,1) \\ 0 = g(4,2) \end{matrix}\right\}$$

implies $y_{t+1} = 0$. Similarly, if we consider $0 = g(5,3) = g(4,1) = g(5,2) = g(6,3)$, we conclude that $w_{t+4} = 0$, $w_{t+5} = 0$ and $y_{t+2} = 0$.

Moreover, $0 = g(2,3)$ and $w_{t+1} \neq 0$, imply

$$y_{t-5} = \frac{(-1)^t w_{t+1}}{(2t-5-2w_{t+1})}.$$

But, by substituting in $g(3,4)$ the values of y_{t-5}, y_{t-6} and $y_{t-k} = 0$ for $k = 2, 3, 4$, we obtain

$$0 = g(3,4) = (-1)^t(-1+2w_{t+1})x_4 + \frac{(8t^2-43t+58-8w_{t+1}t+19w_{t+1})w_{t+1}}{2t-5-2w_{t+1}} + \frac{-2t^3+17t^2-48t+45}{2t-5-2w_{t+1}}.$$

It is easy to check that $-1+2w_{t+1} \neq 0$, whence

$$x_4 = -\frac{1}{2}\frac{-8t^2w_{t+1}+43tw_{t+1}+8tw_{t+1}^2-58w_{t+1}-19w_{t+1}^2+2t^3-17t^2+48t-45}{(-2t+5+2w_{t+1})(-1+2w_{t+1})}.$$

Then, bearing in mind $w_{t+1} \neq 0, -1, y_{t-j} = 0$ for $j = 1, 2, 3, 4$ and the value of y_{t-5}, we have

$$0 = g(4,4) = \frac{(-1)^{t-1}}{6}\frac{(2t-5)(2t-7)w_{t+1}(w_{t+1}+1)(-t+3+3w_{t+1})}{(-2t+5+2w_{t+1})(-1+2w_{t+1})}.$$

Hence,

$$w_{t+1} = \frac{t}{3} - 1.$$

Moreover, from $-2t+5+2w_{t+1} \neq 0$, we obtain $0 \neq -4t+9$. Furthermore, from $0 = g(5,5)$, we deduce

$$x_5 = \frac{1}{8}\frac{t(t-2)(t-3)^2}{(4t-9)(4t-15)}.$$

But, by substituting in $g(6,5) = 0$ the values of y_{t-j} for $j = 0, \ldots, 5$, w_{t+i} for $i = 1, 3, 4, 5$, and x_5, it follows that

$$0 = g(6,5) = \frac{(-1)^{t-1}}{180}\frac{t(t-3)^2(2t-5)(2t-7)(t-2)(2t-3)}{(4t-9)(4t-15)},$$

a contradiction. Consequently, $w_{t+1} + w_{t+1}^2 \equiv 0 \bmod p$. $\square$

Now, by an inductive argument and Lemma 2.4, we generalize Lemma 4.3:

Theorem 4.4. *If $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + t + 6 \leq m - 2c - 1$, then $w_{t+2} = 0$ and $w_{t+1} \in \{0, -1\}$.*

Thus now we consider the cases corresponding to each possible value of w_{t+1} in what follows.

5. The case $w_{t+1} = -1$

In this section, we analyze the case $w_{t+1} \equiv -1 \bmod p$ for $l \geq 3$. For $2l + t + k \leq m - 2c - 1$, we find the values of w_{t+i} for $i = 2, \ldots, k-1$ and y_{t+j} for $j = 0, \ldots, k-2$.

Lemma 5.1. *Suppose that $l \geq 3$, $2l+t+3 \leq m-2c-1$, $(t+2)(t+1)(2t-1)(2t-2l+5) \not\equiv 0 \bmod p$ and $w_{t+1} \equiv -1 \bmod p$. Then,*

(i) $w_{t+2} = 0$.
(ii) $y_{t+1} = \frac{(2t+2)(2t+1)}{(t+2)(t+1)}y_{t-1}$, $y_t = \frac{(2t+1)}{t+1}y_{t-1}$, $y_{t-2} = \frac{1}{2}y_{t-1}$, $y_{t-3} = \frac{1}{2}\frac{t-1}{2t-1}y_{t-1}$, $y_{t-4} = \frac{(t-2)}{4(2t-1)}y_{t-1}$.

Proof. (i) By Lemma 2.5, we obtain $w_{t+2} = 0$.
(ii) Immediate. It is the solution of the linear homogeneous system

$$\left\{\begin{matrix} g(r,i) = 0, & \text{for } 1 \leq r, i \leq 2 \\ g(3,1) = 0. & \end{matrix}\right\}. \quad \square$$

Now, arguing by an inductive argument, we extend the result of the above lemma:

Proposition 5.2. *Suppose that $l = 3$, $2l + t + k \leq m - 2c - 1$, with $k \geq 3$, $w_{t+1} \equiv -1 \bmod p$,*

$$\prod_{j=0}^{k-3}(2t-(2j+1)) \not\equiv 0 \bmod p, \quad \prod_{j=1}^{k-1}(t+j) \not\equiv 0 \bmod p,$$

and

$$\prod_{j=0}^{\left[\frac{k-1}{2}\right]-2} (2t+2j+1) \not\equiv 0 \bmod p.$$

Then,

(i) $w_{t+2} = \cdots = w_{t+k-1} = 0$.

(ii) $y_{t+i} = 2^{\left[\frac{i+1}{2}\right]} y_{t-1} \prod_{j=0}^{\left[\frac{i}{2}\right]} \frac{2t+2j+1}{t+\left[\frac{i-1}{2}\right]+2+j}$, $0 \le i \le k-2$.

(iii) $y_{t+j-1} = \frac{2t+j}{t+j} y_{t+j-2}$ *for* $1 \le j \le k-1$.

Proof. We argue by induction on k. If $k = 3$, it follows from Lemma 5.1.

By an inductive argument, we may assume $w_{t+2} = \cdots = w_{t+k-1} = 0$ and the values y_{t+i} for $i = -4, \ldots, k-2$ given in the statement. We claim that $w_{t+k} = 0$. Indeed,

$$0 = g(k+1, 2) = (-1)^k \frac{tk}{(t+k-2)(t+k-1)} 2^{[\frac{k+1}{2}]-2} \prod_{j=0}^{[\frac{k}{2}]-2} \frac{2t+2j+1}{t+j+\left[\frac{k-1}{2}\right]} y_{t-1} w_{t+k},$$

so if $y_{t-1} \not\equiv 0 \bmod p$, then the last equality implies $w_{t+k} = 0$. If $y_{t-1} \equiv 0 \bmod p$, then, by induction, we have $\frac{\alpha_{4,t+k}}{x_3'} = y_{t+k-6} - 3y_{t+k-5} + 3y_{t+k-4} - y_{t+k-3} = 0$ and $\frac{\alpha_{3,t+k}}{x_3'} = y_{t+k-6} - 2y_{t+k-5} + y_{t+k-4} = 0$, so $0 = g(k+1, 3) = (-1)^{t-1} w_{t+k}$. Hence, $w_{t+k} = 0$. In any case,

$$w_{t+k} = 0.$$

Moreover, from

$$0 = g(k+1, 1) = \binom{t+k-1}{t-1} \left(\frac{2t+k}{t} y_{t+k-2} - \frac{t+k}{t} y_{t+k-1} \right),$$

we obtain

$$y_{t+k-1} = \frac{2t+k}{t+k} y_{t+k-2}.$$

Then, by induction, we conclude that

$$y_{t+k-1} = 2^{\left[\frac{k}{2}\right]} y_{t-1} \prod_{j=0}^{\left[\frac{k-1}{2}\right]} \frac{2t+2j+1}{t+\left[\frac{k-2}{2}\right]+2+j}. \quad \square$$

Now, we extend this result for p-groups G such that $l > 3$. We do it in two steps, because the inductive argument on G does not work properly when we consider y_j.

Theorem 5.3. *If* $l \ge 3$, $l + c_0 \le \frac{p-1}{2}$, $2l + t + k \le m - 2c - 1$, *with* $k \ge 3$ *and* $w_{t+1} \equiv -1 \bmod p$, *then* $w_{t+2} = \cdots = w_{t+k-1} = 0$.

Proof. It follows from Proposition 5.2(i) (for $l = 3$), induction on $l + |G|$ and Lemma 2.4. $\square$

Theorem 5.4. *If* $l \ge 3$, $l + c_0 \le \frac{p-1}{2}$, $2l + t + k \le m - 2c - 1$, *with* $k \ge 3$, *and* $w_{t+1} \equiv -1 \bmod p$, *then* $y_{t+j-1} = \frac{2t+j}{t+j} y_{t+j-2}$ *for* $1 \le j \le k-1$.

Proof. For $l = 3$, the result holds by Proposition 5.2(ii). If $l > 3$, we apply Theorem 5.3 in order to obtain $w_{t+i} = 0$ for $i = 2, \ldots, k-1$. Moreover, for $j = 2, \ldots, k$, we have

$$0 = g(j, 1) = \binom{t+j-2}{t-1} \left(\frac{2t+j-1}{t} y_{t+j-3} - \frac{t+j-1}{t} y_{t+j-2} \right).$$

Thus,

$$y_{t+j-2} = \frac{2t+j-1}{t+j} y_{t+j-3}. \quad \square$$

6. The case $w_{t+1} = 0$

In this section, we analyze the case $w_{t+1} = 0$ for $l \ge 3$. We omit the proofs because the arguments are the same as the ones used in the corresponding results of the former section.

Lemma 6.1. *Suppose that $l \geq 3$, $2l + t + 3 \leq m - 2c - 1$, $w_{t+1} \equiv 0 \bmod p$, $2t - 2l + 3 \not\equiv 0 \bmod p$,*

$$\prod_{j=1}^{2}(2t - (2j+1)) \not\equiv 0 \mod p \quad \textit{and} \quad \prod_{j=0}^{1}(t+j) \not\equiv 0 \bmod p.$$

Then,

(i) $w_{t+2} = 0$.
(ii) $y_{t+1} = 2\frac{(2t-1)}{(t+1)}y_{t-1}$, $y_t = \frac{(2t-1)}{t}y_{t-1}$, $y_{t-2} = \frac{1}{2}y_{t-1}$, $y_{t-3} = \frac{1}{2}\frac{(t-2)}{(2t-3)}y_{t-1}$, $y_{t-4} = \frac{1}{4}\frac{(t-3)}{(2t-3)}y_{t-1}$.

Proposition 6.2. *If $l = 3$, $2l + t + k \leq m - 2c - 1$ with $k > 3$, $w_{t+1} \equiv 0 \bmod p$,*

$$\prod_{j=1}^{k-1}(2t - (2j+1)) \not\equiv 0 \bmod p \quad \textit{and} \quad \prod_{j=0}^{k-2}(t+j) \not\equiv 0 \bmod p,$$

then

(i) $w_{t+2} = \cdots = w_{t+k-1} = 0$.
(ii) $y_{t+i} = 2^{\left[\frac{i+1}{2}\right]}y_{t-1}\prod_{j=0}^{\left[\frac{i}{2}\right]}\frac{2t+2j-1}{t+\left[\frac{i-1}{2}\right]+j+1}$ *for* $1 \leq i \leq k-2$ *and* $y_t = \frac{(2t-1)}{t}y_{t-1}$.

Moreover, $y_{t+j} = \frac{2t+j-1}{t+j}y_{t+j-1}$ *for* $j = 1, \ldots, k-2$.

Theorem 6.3. *If $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$, $2l + t + k \leq m - 2c - 1$ and $w_{t+1} \equiv 0 \bmod p$, then $w_{t+2} = \cdots = w_{t+k-1} = 0$.*

Theorem 6.4. *If $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$, $2l + t + k \leq m - 2c - 1$ and $w_{t+1} \equiv 0 \bmod p$, then $y_{t+j} = \frac{2t+j-1}{t+j}y_{t+j-1}$ for $j = 1, \ldots, k-2$.*

7. Proof of the Main Theorem

In this section, we prove the Main Theorem. Arguing by contradiction, we assume that $2l + c_0 + 3 \leq m - 2c - 1$. If we prove that $2l + c_0 + 3 \leq m - 2c - 1$ and $l \geq 3$ implies $t > c_0 + 1$, then, by using Lemma 2.1, we derive a contradiction, that shows the Main Theorem. In order to check that $t > c_0 + 1$, we consider separately the cases $t < c_0 - 2$, $t = c_0 - i$ with $i = 0, 1, 2$ and $t = c_0 + 1$. This analysis is based on the conclusions that we have found in the previous sections.

Proposition 7.1. *If $p > 29$, $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + c_0 + 3 \leq m - 2c - 1$, then $t > c_0 - 3$.*

Proof. Arguing by contradiction, we assume that $t \leq c_0 - 3$. Then, $2l + t + 6 \leq 2l + c_0 - 3 + 6 = 2l + c_0 + 3 \leq m - 2c - 1$ and, by Theorem 4.4, it follows that $w_{t+2} = 0$ and $w_{t+1} = 0, -1$. If $w_{t+1} = 0$, from Theorem 6.3 with $t + k = c_0 + 3$, we deduce that $w_i = 0$ for $i = 2, \ldots, c_0 + 2$ and, from Theorem 6.4,

$$y_{t+j} = \frac{2t + j - 1}{t + j}y_{t+j-1}, \quad \text{for } j = 1, \ldots, c_0 + 1 - t.$$

But, $y_{t+c_0+1-t} = y_{c_0+1} = w_1 = 0$, so $y_{t+j} = 0$ for $j = 0, \ldots, c_0 + 1 - t$. Moreover, from Lemma 6.1, it follows that $y_{t-i} = 0$ for $i = 1, \ldots, 4$. Furthermore, as $0 = w_1 = \cdots = w_{t-1}$, we know that $y_{c_0+1+j} = 0$ for $1 \leq j \leq t - 2$. Hence,

$$0 = y_{t-4} = \cdots = y_{c_0+t-1}.$$

If we take $s = t - 4$, we have

$$2s - 1 = 2t - 9 \leq (c_0 - 3) + t - 9 = c_0 + t - 12 \leq c_0 + t - 1,$$

and, by Lemma 2.3, we conclude that either there exists j such that $1 \leq j \leq t - 4$ and

$$l + t - 4 - j \equiv 0 \bmod p,$$

or there exists w such that $2 \leq w \leq 2t - 10$ and

$$2l + 2t - 9 - w \equiv 0 \bmod p.$$

But if $1 \leq j \leq t - 4$, then

$$3 \leq l \leq l + t - 4 - j \leq l + t - 5 \leq l + c_0 - 3 - 5 \leq l + c_0 - 8 \leq \frac{p-1}{2} - 8 < p, \quad \text{if } p > 29$$

and if $2 \leq w \leq 2t - 10$, then

$$7 \leq 2l + 1 \leq 2l + 2t - 9 - w \leq 2l + 2t - 11 \leq 2l + 2c_0 - 19 \leq p - 22.$$

That is, $w_{t+1} = 0$ is not possible. Similarly, $w_{t+1} = -1$ is refuted by considering the corresponding theorems of Section 5. □

Now, the cases $t = c_0 - i$ with $i = -1, 0, 1, 2$ remain to be studied. In order to disprove them, we collect the information that we have obtained in Sections 2–4 in the following three lemmas.

Lemma 7.2. *If $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + t + 3 \leq m - 2c - 1$, then one of the following statements is satisfied:*

(i) $w_{t+1} + w_{t+1}^2 = 0$ *and* $w_{t+2} = 0$.
(ii) $y_{t-i} = 0$ *for* $i = 0, \ldots, 2l - 4$.

Proof. Immediate. It is enough to apply Lemma 2.5, Proposition 3.2 and Lemma 4.2. □

If (ii) and $2l + t + 4 \leq m - 2c - 1$ hold, we calculate the value of y_{t+1}:

Lemma 7.3. *If $p > 2$, $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + t + 4 \leq m - 2c - 1$ and $y_{t-i} = 0$ for $i = 0, \ldots, 2l - 4$, then $y_{t+1} = 0$ holds.*

Proof. Suppose that $y_{t+1} \neq 0$. Then, the system

$$\left\{\begin{array}{ll} 0 = y_{t-i}, & \text{for } i = 0, \ldots, 2l-4, \\ 0 = g(3, 1) & \\ 0 = g(1, l-1) & \\ 0 = g(4, 2) & \end{array}\right\}$$

implies

$$w_{t+2} = \frac{1}{2}(t+1)(-t + 2w_{t+1}),$$
$$y_{t-2l+2} = \frac{(2l-3)}{2} y_{t-2l+3},$$
$$w_{t+3} = \frac{1}{6}(t+1)(t+2)(-2t + 3w_{t+1}).$$

If $y_{t-2l+3} \neq 0$, we derive a contradiction, considering the system

$$\left\{\begin{array}{l} 0 = g(2, l-1) \\ 0 = g(3, l) \\ 0 = g(4, l) \end{array}\right\}.$$

Thus, $y_{t-2l+3} = 0$. This fact jointly with $0 = g(3, l)$ implies $y_{t-2l+2} = 0$, $w_{t+2} = 0$ and $w_{t+1} = \frac{t}{2}$. However, by substituting these values in $g(4, l)$, we derive a contradiction. □

A similar argument shows:

Lemma 7.4. *If $p > 2$, $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$, $2l + t + 5 \leq m - 2c - 1$ and $y_{t-i} = 0$ for $i = 0, \ldots, 2l - 4$, then $y_{t+2} = 0$.*

Proof. From Lemma 7.3, we know that $y_{t+1} = 0$. From $g(1, l-1) = 0$, we deduce that

$$y_{t-2l+2} = \frac{(2l-3)}{2} y_{t-2l+3}.$$

In addition, from $y_{t-i} = 0$ for $i = 0, \ldots, 2l - 4$, it follows that

$$0 = g(5, l) = (-1)^{t-1} w_{t+4}.$$

Hence, $w_{t+4} = 0$. Now, we assume, by contradiction, that $y_{t+2} \neq 0$. Then, from the system $0 = g(4, 1) = g(5, 2)$, we deduce that

$$w_{t+2} = -\frac{1}{4}(t^2 + t) + \frac{2}{3}(t+1)w_{t+1} \quad \text{and} \quad w_{t+3} = -\frac{1}{12}(t+2)(t+1)(t - 2w_{t+1}).$$

An argument similar to the proof of Lemma 7.3 completes the proof. □

Now, we disprove $t = c_0 - i$ for $i = 0, \ldots, 2$ in the following proposition.

Proposition 7.5. *If $p > 29$, $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + c_0 + 3 \leq m - 2c - 1$, then $t > c_0$.*

Proof. From Proposition 7.1, we know that $t > c_0 - 3$. Suppose that $t = c_0 - i$, with $i \in \{0, 1, 2\}$.

If Lemma 7.2(i) holds, then from Lemmas 5.1 and 6.1, according to $w_{t+1} = -1, 0$, we can write $y_{t-j} = a_j y_{t-1}$ for $j = -1, \ldots, 4$, where a_j is defined in Lemma 5.1 and Lemma 6.1, respectively. Now, set $k = c_0 + 3 - t$ and we apply Theorem 5.4 if $w_{t+1} = -1$ or Theorem 6.4 if $w_{t+1} = 0$ to conclude that $0 = y_{c_0+1}$. Then,

$$y_{t-4} = \cdots = y_{c_0} = 0.$$

Moreover, $w_1 = \cdots = w_{t-1} = 0$, so

$$y_{t-4} = \cdots = y_{t+c_0-1} = 0.$$

Now by applying Lemma 2.3, we derive a contradiction.

If Lemma 7.2(ii) holds, then:

(a) We disprove the case $t = c_0$ by Lemma 2.3 with $s = c_0$.
(b) If $t = c_0 - 1$, it follows that

$$0 = y_{c_0-2l+3} = \cdots = y_{c_0-1} = y_{c_0+1} = \cdots = y_{2c_0-2}.$$

and, by Lemma 7.3, it follows that $y_{c_0} = 0$, so we set $s = c_0 - 1$ in Lemma 2.3 in order to derive a contradiction.
(c) If $t = c_0 - 2$, we argue like in (a) and (b) after applying Lemmas 7.3 and 7.4. □

It only remains to prove that $t \neq c_0 + 1$. The argument for disproving $t = c_0 + 1$ is different because the results for which the hypothesis is $2l + t + k \leq m - 2c - 1$ with $k \geq 3$ cannot be used.

Proposition 7.6. *If $l \geq 3$, $l + c_0 \leq \frac{p-1}{2}$ and $2l + c_0 + 3 \leq m - 2c - 1$, then $t \neq c_0 + 1$.*

Proof. Suppose $t = c_0 + 1$. Then,

$$0 = y_{c_0+1} = \cdots = y_{2c_0}.$$

Moreover, $y_t = y_{c_0+1} = w_1 = 0$ implies

$$0 = g(2, 1) = y_{c_0}(2c_0 - 2w_{c_0+2} + 1).$$

But, from Lemma 2.3, with $s = c_0$, we conclude that $y_{c_0} \neq 0$ and $w_{c_0+2} = \frac{2c_0+1}{2}$. Furthermore, from

$$0 = f(1, 2, 2l + c_0 - 1) = (-1)^{c_0+1} w'_{c_0+1}(\alpha_{2,2l+c_0-1} + \alpha_{1,2l+c_0-1})$$

it follows that $\alpha_{2,2l+c_0-1} = -\alpha_{1,2l+c_0-1}$, and by (P3),

$$\alpha_{1,2l+c_0} = 2\alpha_{1,2l+c_0-1}.$$

We write $w'_i = 0$, for $i = 1, \ldots, c_0$ in terms of x'_i by using that $\binom{j-k-1}{i+j-2k-1} = \binom{j-k-1}{k-i}$ and the formula (1). Then, we obtain the following homogeneous linear system:

$$\sum_{k=i}^{\frac{i+2l+c_0}{2}} (-1)^{k-i} \binom{2l + c_0 - k}{2l + c_0 + i - 2k} x'_k = 0, \tag{16}$$

where the variables are $x'_1, \ldots, x'_{c_0+l}$. But $x'_i = 0$ for $i = 1, \ldots, l-1$, whence the system has c_0 equations in $c_0 + 1$ variables.

We add a new variable $s = y'_{c_0-1}$. Then, we have the following equations:

$$\left\{ \begin{matrix} \alpha_{1,2l+c_0} - 2s = 0 \\ \alpha_{1,2l+c_0-1} - s = 0 \end{matrix} \right\},$$

that is,

$$\left\{ \begin{matrix} \alpha_{1,2l+c_0} - 2s = \displaystyle\sum_{k=1}^{\left[\frac{2l+c_0}{2}\right]} (-1)^{k-1} \binom{2l + c_0 - k - 1}{2l + c_0 - 2k} x'_k - 2s = 0 \\ \alpha_{1,2l+c_0-1} - s = \displaystyle\sum_{k=1}^{\left[\frac{2l+c_0-1}{2}\right]} (-1)^{k-1} \binom{2l + c_0 - k - 2}{2l + c_0 - 2k - 1} x'_k - s = 0 \end{matrix} \right\}. \tag{17}$$

These two last equations jointly with the system (16) form a homogeneous linear system of $c_0 + 2$ equations in the $c_0 + 2$ variables $x'_l, \ldots, x'_{l+c_0}, s$. In the equations of (16) and (17), we make the following changes:

$$k = l - 1 + u, \qquad (-1)^k x'_k = \gamma_k,$$

in order to obtain

$$\left\{ \begin{matrix} (-1)^i \displaystyle\sum_{u=\max\{l,i-l+1\}}^{\left[\frac{c_0+i}{2}\right]+1} \binom{l + c_0 + 1 - u}{c_0 + i + 2 - 2u} \gamma_u = 0, \quad i = 1, \ldots, c_0 \\ (-1) \displaystyle\sum_{u=l}^{\left[\frac{c_0}{2}\right]+1} \binom{l + c_0 - u}{c_0 + 2 - 2u} \gamma_u - 2s = 0 \\ (-1) \displaystyle\sum_{u=l}^{\left[\frac{c_0-1}{2}\right]+1} \binom{l + c_0 - 1 - u}{c_0 + 1 - 2u} \gamma_u - s = 0 \end{matrix} \right\}. \tag{18}$$

If we remove $(-1)^i$ in the c_0 first equations and we multiply the last two equations by (-1), we obtain a homogeneous linear system whose system matrix of order $(c_0+2)\times(c_0+2)$ is defined by $A=(a_{i,j})$ where

$$a_{i,j}=\begin{cases}\binom{l+c_0+1-j}{c_0+2+i-2j}, & \text{if } 1\le i\le c_0 \text{ and } 1\le j\le c_0+1;\\ 0, & \text{if } 1\le i\le c_0 \text{ and } j=c_0+2;\\ \binom{l+c_0-j}{c_0+2-2j}, & \text{if } i=c_0+1 \text{ and } 1\le j\le c_0+1;\\ 2, & \text{if } i=c_0+1 \text{ and } j=c_0+2;\\ \binom{l+c_0-1-j}{c_0+1-2j}, & \text{if } i=c_0+2 \text{ and } 1\le j\le c_0+1;\\ 1, & \text{if } i=c_0+2 \text{ and } j=c_0+2.\end{cases}$$

Then, the determinant of A is given by

$$\det(A)=-2\det(A_1)+\det(A_2),$$

where $A_1=(b_{i,j})$ with

$$b_{i,j}=\begin{cases}a_{i,j}, & \text{if } 1\le i\le c_0,\ 1\le j\le c_0+1;\\ a_{c_0+2,j}, & \text{if } i=c_0+1,\ 1\le j\le c_0+1;\end{cases}$$

and A_2 is the main submatrix of A of order $(c_0+1)\times(c_0+1)$. Now if we define $x=l+c_0+1$, $n=c_0+1$ and $\lambda=1,2$, the determinants of A_1 and A_2 are as computed in Lemma 2.7. Then,

$$\det(A)=\frac{(-1)^{c_0}}{|F(1)|}F(l+c_0+1)T(l+c_0+1),$$

where

$$T(x)=\frac{2x-2c_0^2-3c_0-3}{2x-c_0-3},$$

and

$$F(x)=\prod_{i=2}^{\left[\frac{n}{2}\right]+1}(x-i)^{i-1}\prod_{i=\left[\frac{n}{2}\right]+2}^{n}(x-i)^{n+1-i}\prod_{i=2}^{\left[\frac{n-1}{2}\right]+1}(2x-2i-1)^{i-1}\prod_{i=\left[\frac{n-1}{2}\right]+2}^{n-1}(2x-2i-1)^{n-i}.$$

But if c_0 is even, the power of the factor $2x-c_0-3$ is $c_0/2$. Then, for c_0 even, $\frac{F(x)}{|F(1)|2x-c_0-3}$, with $x=l+c_0+1$, is not 0 modulo p when $l+c_0\le\frac{p-1}{2}$ and $\det(A)$ is 0 modulo p if and only if $2x-2c_0^2-3c_0-3$, with $x=l+c_0+1$, is 0 modulo p. If c_0 is odd, then the same happens. As the system (18) has a non-trivial solution, it follows that $\det(A)$ is 0 modulo p and, consequently,

$$2x-2c_0^2-3c_0-3=2l-2c_0^2-c_0-1\equiv 0 \bmod p.$$

Moreover, we know that

$$0=g(2,2)=\frac{1}{2}y_{c_0}-\frac{5}{2}y_{c_0-1}+2y_{c_0-2}$$

and, because of $y'_{c_0}=2y'_{c_0-1}$, it follows that

$$y'_{c_0-2}-\frac{3}{4}y'_{c_0-1}=0.$$

Then, we consider the system

$$\left\{\begin{array}{l}w'_i=0,\quad i=1,\dots,c_0\\ y'_{c_0-1}-s=0\\ y'_{c_0-2}-\frac{3}{4}s=0,\end{array}\right\}$$

where $s = y'_{c_0-1}$. We notice that $y'_{c_0-2} = \sum_{k=1}^{\left[\frac{2l+c_0-1}{2}\right]} (-1)^{k-1} \binom{2l+c_0-2-k-1}{2l+c_0-1-2k-1} x'_k$. Therefore if we make the same changes as in the other system, we obtain

$$\left\{\begin{array}{l} 0 = (-1)^i \displaystyle\sum_{u=\max\{l,i-l+1\}}^{\left[\frac{c_0+i}{2}\right]+1} \binom{l+c_0+1-u}{c_0+i+2-2u} \gamma_u, \quad i = 1, \ldots, c_0 \\ 0 = (-1) \displaystyle\sum_{u=l}^{\left[\frac{c_0-1}{2}\right]+1} \binom{l+c_0-1-u}{c_0+1-2u} \gamma_u - s \\ 0 = (-1) \displaystyle\sum_{u=l}^{\left[\frac{c_0-1}{2}\right]+1} \binom{l+c_0-2-u}{c_0-2u} \gamma_u - \frac{3}{4} s \end{array}\right\}. \tag{19}$$

We multiply the first c_0 equations of (19) by $(-1)^i$ and the last two equations by (-1). Then, the system matrix obtained is $B = (b_{i,j})$, where

$$b_{i,j} = \begin{cases} \binom{l+c_0+1-j}{c_0+i+2-2j}, & \text{if } 1 \le i \le c_0 \text{ and } 1 \le j \le c_0+1; \\ 0, & \text{if } 1 \le i \le c_0 \text{ and } j = c_0+2; \\ \binom{l+c_0-1-j}{c_0+1-2j}, & \text{if } i = c_0+1 \text{ and } 1 \le j \le c_0+1; \\ 1, & \text{if } i = c_0+1 \text{ and } j = c_0+2; \\ \binom{l+c_0-2-j}{c_0-2j}, & \text{if } i = c_0+2 \text{ and } 1 \le j \le c_0+1; \\ \frac{3}{4} & \text{if } i = c_0+2 \text{ and } j = c_0+2. \end{cases}$$

Then,

$$\det(B) = -\det(B_1) + \frac{3}{4}\det(B_2),$$

where B_1 is the submatrix formed from the first c_0 rows of B and row c_0+2 with the first c_0+1 columns and B_2 is the main submatrix of order $(c_0+1) \times (c_0+1)$. Then if we set $x = l + c_0 + 1$, $n = (c_0+1)$ and $\lambda = 2, 3$ in Lemma 2.7, we conclude that

$$\det(B) = (-1)^{c_0} \frac{H(x)}{|H(1)|} \frac{c_0}{2} \left(-(c_0-1) + \frac{3}{2} c_0 \frac{T_1(x)}{|T_1(1)|} \right),$$

where

$$T_1(x) = \begin{cases} x - \frac{c_0}{2} - 2, & \text{if } c_0 \text{ is even;} \\ \frac{2x-c_0-5}{2}, & \text{if } c_0 \text{ is odd;} \end{cases}$$

and

$$H(x) = H_1(x) H_2(x)$$

$$H_1(x) = \prod_{i=2}^{\left[\frac{n-1}{2}\right]+1} (x-i)^{i-1} \prod_{i=\left[\frac{n-1}{2}\right]+2}^{n} (x-i)^{n+1-i} \prod_{i=2}^{\left[\frac{n-1}{2}\right]+1} (2x-2i-1)^{i-1} \prod_{i=\left[\frac{n-1}{2}\right]+2}^{n-1} (2x-2i-1)^{n-i},$$

$$H_2(x) = \begin{cases} \frac{1}{\left(x-\left(\left[\frac{n-1}{2}\right]+2\right)\right)\left(2x-2\left[\frac{n-1}{2}\right]-3\right)\left(2x-2\left[\frac{n-1}{2}\right]-5\right)}, & \text{if } n \text{ is even;} \\ \frac{1}{\left(x-\left(\left[\frac{n-1}{2}\right]+2\right)\right)\left(2x-2\left[\frac{n-1}{2}\right]-3\right)}, & \text{if } n \text{ is odd.} \end{cases}$$

But this means that

$$\det(B) = (-1)^{c_0} \frac{H(l+c_0+1)}{|H(1)|} \frac{c_0}{4(c_0+2)} (6l - 2c_0^2 + c_0 - 2)$$

and as $l + c_0 \le \frac{p-1}{2}$, we have

$$\det(B) \equiv 0 \bmod p$$

if and only if

$$6l - 2c_0^2 + c_0 - 2 \equiv 0 \bmod p.$$

But $2l - (2c_0^2 + c_0 + 1) \equiv 0 \bmod p$, so that

$$6l - 2c_0^2 + c_0 - 2 = 3(2l - (2c_0^2 + c_0 + 1)) - (2c_0 + 1)^2 \equiv -(2c_0 + 1)^2 \bmod p.$$

Therefore $\det(B) \not\equiv 0 \bmod p$, a contradiction. □

Remark. The results of this section show the Main Theorem for $p > 29$. For $p \le 29$ if we consider the Jacobi identities $f(i, j, k) = 0$, where $i + j + k \le (2l + c_0 + 3) \le m - 2c - 1$, that factor into a product of polynomials and we analyze all the possibilities, we conclude that:

1. $2c \ge m - (2l + c_0 + 2)$ if $p \le 29$, $l \ge 3$ and $l + c_0 \le \frac{p-3}{2}$ except for $p = 29$, $l = 3$ and $c_0 = 10$.
2. $2c \ge m - (2l + c_0 + 1)$ if $p \le 29$ and $l + c_0 = \frac{p-1}{2}$, $l \ge 3$, except for $p = 29$, $l = 3$ and $c_0 = 11$.
3. $2c \ge m - (2l + c_0 + 3)$ if $p = 29$, $l = 3$ and $c_0 = 10$.
4. $2c \ge m - (2l + c_0 + 2)$ if $p = 29$, $l = 3$ and $c_0 = 11$.

In order to obtain these bounds, we have designed an algorithm that checks all possibilities and obtains whether there exists the corresponding Lie algebra for attaining the bound. We have implemented this algorithm in MAPLE.

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